A different view on controllability and observability of continuous time linear systems with interval coefficients

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Abstract

We discuss the controllability and observability of time-invariant (continuous time) linear systems with interval coefficients using the notion of being full rank of interval matrices. The most important advantage of the proposed attitude is to consider these two essential concepts, that is, controllability and observability, in interval time-invariant linear systems, which, in turn, may play important roles in the analysis of uncertain systems. Some different definitions on to be full rank of matrices have been utilized to propose different views on the controllability and observability of interval linear systems according to different criteria. Finally, in several control-observation processes, the controllability and observability are evaluated based on the given achievements.

1 Introduction

The importance of the control issue is clear, given its role in automating processes and enhancing efficiency. In a control system, there are three types of

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variables: State variables, control variables, and output variables. Whenever in a control system, there is the possibility of determination of control variable, that the system can be transferred from the initial state to a terminal state; then the control system has the controllability property. If the control variable is influenced by the output variable, the control system has the observability property.

Past works have been largely devoted to obtaining various criteria of controllability and observability. The concept of structural controllability has been studied in [10, 13, 5]. Lin et al. [9] expressed the concept of C-controllability and C-observability for uncertain descriptor systems with interval perturbations in all matrices. Cheng and Zhang [2] checked the robust controllability for a class of uncertain linear time-invariant multi-input multi-output (MIMO) systems, and for this purpose, the single-sign constraint and the feedback constraint have been considered. Also, Chou, Chen, and Zhang [3] extended the method presented in [2] for linear uncertain descriptor systems. Ismail and Bandyopadhyay [8] obtained a sufficient condition for determining the controllability and observability of linear symmetric interval systems. Wang and Michel [14] established necessary and sufficient conditions for the controllability of single-input multi-output linear time-invariant system with interval plants in state matrix and for the observability of multi-input single-output linear time-invariant system with interval plants in state matrix. Yang [15] extended the theory in [14] to systems with interval plants for both state and control matrices (or state and output matrices) with single-input single-output.

In this paper, the controllability and observability of MIMO time-invariant linear systems whose all coefficients can be intervals, are considered, while there are no additional constraints, for example, symmetry of the state matrix, for these systems. Necessary and sufficient conditions of the controllability and observability of time-invariant (continuous time) linear systems with interval coefficients in all matrices, are discussed. The controllability and observability of the systems are investigated using the fullness of the rank of the interval matrices. Because of computational considerations, some criteria on to be full rank of matrices which have been presented by Shary in [12] are applied to define different criteria for the controllability and observability of interval linear control systems.

The structure of the paper in the following sections is as follows. In Section 2, the time-invariant linear system with interval coefficients is introduced. In Section 3, the controllability and observability are considered. In Section 4, criteria-based concepts of controllability and observability of MIMO time-invariant linear system with interval coefficients and full rank interval matrices have been given. In Section 5, numerical results to evaluate the given discussions are stated.

2 Preliminaries and statement of the problem

In this section, at first, the interval parameter transformation, which is used for analyzing the necessary and sufficient conditions in interval optimal control problems in [7, 11, 6], is mentioned. Consider the closed bounded interval $\mathsf{S} = [s^0, s^1] = \{x \in \mathbb{R}|s^0 \leqslant x \leqslant s^1\}$ for $s^0, s^1 \in \mathbb{R}$. Any value in S may be stated as $s(\lambda) = s^0 + \lambda(s^1 - s^0), 0 \leqslant \lambda \leqslant 1$, and also the left and right endpoints of the interval $\mathsf{S} = [s^0, s^1]$ may be displayed as $s^0 = \min s(\lambda), s^1 = \max s(\lambda), 0 \leqslant \lambda \leqslant 1$; see [1].
Suppose that $\mathsf{S} = [s^0, s^1] = \{s(\lambda_1)|\lambda_1 \in [0,1]\}$ and $\mathsf{Y} = [y^0, y^1] = \{y(\lambda_2)|\lambda_2 \in [0,1]\}$ are two closed bounded intervals. The algebraic operations of intervals are presented with respect to parameters as follows, [1]:

$$\mathsf{S} \oplus \mathsf{Y} = \{s(\lambda_1) + y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\},$$

$$\mathsf{S} \ominus \mathsf{Y} = \{s(\lambda_1) - y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\},$$

$$\mathsf{S} \odot \mathsf{Y} = \{s(\lambda_1).y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\},$$

$$k\mathsf{S} = \{ks(\lambda)|\lambda \in [0,1]\},$$

$$\mathsf{S} \oslash \mathsf{Y} = \{s(\lambda_1)/y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1], y(\lambda_2) \nvDash 0\}.$$

Definition 1. The matrix $\mathcal{S}$ is called an interval matrix if and only if all of its elements are closed bounded intervals.

Consider the following notations:
$I(\mathbb{R})$ =The set of all closed intervals in $\mathbb{R}$.
$I(\mathbb{R})^n$ =The product space $I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$.
$I(\mathbb{R})^{m \times n}$ =The set of all interval matrices $\mathcal{S}$ with m rows and n columns.
$J[0,1]^{m \times n}$ = The set of all real matrices with m rows and n columns such that all elements of these matrices belong to $[0,1]$.

Definition 2. Let $S = [s_{ij}]_{m \times n}$ be a real matrix and let $\mathcal{S} = [\mathsf{S}_{ij}]_{m \times n} \in I(\mathbb{R})^{m \times n}$ be an interval matrix, where $\mathsf{S}_{ij} = [s_{ij}^0, s_{ij}^1]$. Then S is considered as a member of $\mathcal{S}$ if and only if $s_{ij} \in \mathsf{S}_{ij}$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Proposition 1. An interval matrix $\mathcal{S}$ can be presented by an infinite set of real matrices, that is,
$\mathcal{S} = \{S_\Lambda | S_\Lambda = [s_{ij}(\lambda_{ij})]_{m \times n}, \Lambda = [\lambda_{ij}]_{m \times n} \in J[0,1]^{m \times n}, s_{ij}(\lambda_{ij}) = s_{ij}^0 + \lambda_{ij}(s_{ij}^1 - s_{ij}^0), i = 1, \ldots, m, j = 1, \ldots, n\}$.

Proof. Let $S \in \mathcal{S}$ and let $S = [s_{ij}]_{m \times n}$. Then $s_{ij} \in \mathsf{S}_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$. There exists $\lambda_{ij} \in [0,1]$ such that $s_{ij} = s_{ij}^0 + \lambda_{ij}(s_{ij}^1 - s_{ij}^0)$. In this case, for each element of the matrix S, there exists a real number $\lambda_{ij} \in [0,1], i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. Consider the matrix

$\Lambda = [\lambda_{ij}]_{m \times n}$, $\lambda_{ij} \in [0,1]$, and suppose $s_{ij} = s_{ij}^0 + \lambda_{ij}(s_{ij}^1 - s_{ij}^0) = s_{ij}(\lambda_{ij})$ for $i = 1, \ldots, m, j = 1, \ldots, n$. Then, $S = S_\Lambda$. Therefore, the real matrix S is a member of the infinite set of real matrices.

Now if a real matrix $S = [s_{ij}]_{m \times n}$ belongs to the infinite set of real matrices, then, there is a real matrix $\Lambda = [\lambda_{ij}]_{m \times n}$, $lambda_{ij} \in [0,1]$ such that $S = S_\Lambda$ and $s_{ij} = s_{ij}(\lambda_{ij}) = s_{ij}^0 + \lambda_{ij}(s_{ij}^1 - s_{ij}^0)$, $\lambda_{ij} \in [0,1]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then $s_{ij} \in \mathsf{S}_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and so $S \in \mathcal{S}$.   □

The most important advantage of denoting an interval matrix in the above proposition is that how one can show all real matrices that are included within interval matrix.

Definition 3. An interval matrix $\mathcal{S}$ is called full rank if and only if for each matrix $\Lambda \in J[0,1]^{m \times n}$, the real matrix $S_\Lambda \in \mathcal{S}$ is full rank.

Consider the time-invariant linear system with interval coefficients

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) = \mathcal{C}x(t) + \mathcal{D}u(t), \end{cases} \tag{1}$$

where $\mathcal{A} \in I(\mathbb{R})^{n \times n}$, $\mathcal{B} \in I(\mathbb{R})^{n \times m}$, $\mathcal{C} \in I(\mathbb{R})^{p \times n}$, and $\mathcal{D} \in I(\mathbb{R})^{p \times m}$ are interval matrices that they are denoted according to the Proposition 1 as follows:

$\mathcal{A} = \{A_\Lambda | A_\Lambda = [a_{ij}(\lambda_{ij})], \ \Lambda = [\lambda_{ij}] \in J[0,1]^{n \times n}, a_{ij}(\lambda_{ij}) = a_{ij}^0 + \lambda_{ij}(a_{ij}^1 - a_{ij}^0)\}$,

$\mathcal{B} = \{B_\Gamma | B_\Gamma = [b_{ir}(\gamma_{ir})]_{n \times m}, \ \Gamma = [\gamma_{ir}] \in J[0,1]^{n \times m}, \ b_{ir}(\gamma_{ir}) = b_{ir}^0 + \gamma_{ir}(b_{ir}^1 - b_{ir}^0)\}$,

$\mathcal{C} = \{C_\Theta | C_\Theta = [c_{ki}(\theta_{ki})]_{p \times n}, \ \Theta = [\theta_{ki}] \in J[0,1]^{p \times n}, c_{ki}(\theta_{ki}) = c_{ki}^0 + \theta_{ki}(c_{ki}^1 - c_{ki}^0)\}$,

$\mathcal{D} = \{D_\Omega | D_\Omega = [d_{kr}(\omega_{kr})]_{p \times m}, \ \Omega = [\omega_{kr}] \in J[0,1]^{p \times m}, \ d_{kr}(\omega_{kr}) = d_{kr}^0 + \omega_{kr}(d_{kr}^1 - d_{kr}^0)\}$, $i = 1, \ldots, n$, $\quad j = 1, \ldots, n$, $\quad r = 1, \ldots, m$, $\quad k = 1, \ldots, p$.

In system (1), $u(\cdot)$ and $y(\cdot)$ denote the control and the observation functions, respectively. The first equation describes the process of control and the second equation describes the process of observation. For all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$ from system (1), it can be obtained a system with real coefficients as follows:

$$\begin{cases} \dot{x}(t) = A_\Lambda x(t) + B_\Gamma u(t), \\ y(t) = C_\Theta x(t) + D_\Omega u(t). \end{cases} \tag{2}$$

The first equation of system (2), for all matrices $\Lambda \in J[0,1]^{n \times n}$, $\Gamma \in J[0,1]^{n \times m}$ and with an initial state $x(t_0)$ has the unique solution as, [4],

$$x(t, \Lambda, \Gamma) = \exp[A_\Lambda(t - t_0)]x(t_0) + \int_{t_0}^t \exp[A_\Lambda(t - s)]B_\Gamma u(s)ds.$$

3 Controllability and observability of the time-invariant linear system with interval coefficients

In this section, the concepts of the controllability and observability of system (1) is defined using its parametric form.

Definition 4. System (1) is controllable if and only if system (2) is controllable for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$.

Definition 5. For arbitrary matrices $\Lambda \in J[0,1]^{n \times n}$, $\Gamma \in J[0,1]^{n \times m}$, $\Theta \in J[0,1]^{p \times n}$, $\Omega \in J[0,1]^{p \times m}$, system (2) is controllable if and only if for every initial state $x(t_0) \in \mathbb{R}^n$, the state vector x can be carried from the initial state to any other position as $x(t_1, \Lambda, \Gamma) \in \mathbb{R}^n$, in a finite amount of time and by a specific control u. In other words, system (2) is controllable if and only if t_0 is an arbitrarily initial time, there exists a time as $t_1 \geq t_0$ such that, $x(t_0)$ and $x(t_1, \Lambda, \Gamma)$ to be desired states and the integral equation

$$x(t_1, \Lambda, \Gamma) = \exp[A_\Lambda(t_1 - t_0)]x(t_0) + \int_{t_0}^{t_1} \exp[A_\Lambda(t_1 - s)]B_\Gamma u(s)ds$$

has a unique solution as a control function u.

System (2) is the time-invariant linear system that has been expressed in [4].

Proposition 2. System (1) is controllable if and only if the interval compound matrix

$$\mathcal{M}_{\mathcal{AB}} = \begin{bmatrix} \mathcal{B} & \mathcal{AB} & \mathcal{A}^2\mathcal{B} & \cdots & \mathcal{A}^{n-1}\mathcal{B} \end{bmatrix}$$

is full rank.

Proof. If system (1) is controllable, then system (2) is controllable for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$. Hence the matrix

$$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} B_\Gamma & A_\Lambda B_\Gamma & A_\Lambda^2 B_\Gamma & \cdots & A_\Lambda^{n-1} B_\Gamma, \end{bmatrix}$$

is full rank for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}$ [4]. Therefore, the interval matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank.
Suppose the interval compound matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank. Then, the matrix $M_{A_\Lambda B_\Gamma}$ is full rank for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}$, and system (2) is controllable for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$ [4]. Therefore, system (1) is controllable. $\square$

Definition 6. The linear system (2) has the observability property on an interval time (t_0, t_1) if and only if for arbitrary matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$, any input-output pair $(u(t), y(t))$,

$t_0 \le t \le t_1$ in system (2) uniquely determines the initial state $x(t_0)$.

Definition 7. For arbitrary matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$, the linear system (2) is said to be observable at an initial time t_0 if and only if it has observability property on the interval time (t_0, t_1) where $t_1 \ge t_0$. It is said to be observable if and only if it is observable at every initial time t_0.

Definition 8. System (1) is observable if and only if for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$, system (2) is observable.

Proposition 3. System (1) is observable if and only if the interval compound matrix

$$\mathcal{N}_{\mathcal{C}\mathcal{A}} = \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \mathcal{C}\mathcal{A}^2 \\ \vdots \\ \mathcal{C}\mathcal{A}^{n-1} \end{bmatrix}$$

is full rank.

Proof. If system (1) is observable, then system (2) is observable for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$. So the matrix

$$N_{C_\Theta A_\Lambda} = \begin{bmatrix} C_\Theta \\ C_\Theta A_\Lambda \\ C_\Theta A_\Lambda^2 \\ \vdots \\ C_\Theta A_\Lambda^{n-1} \end{bmatrix}$$

is full rank for all matrices $\Theta \in J[0,1]^{p \times n}$, $\Lambda \in J[0,1]^{n \times n}$, [4]. Therefore, the interval matrix $\mathcal{N}_{\mathcal{C}\mathcal{A}}$ is full rank.

Suppose the interval compound matrix $\mathcal{N}_{\mathcal{C}\mathcal{A}}$ is full rank. Then, the matrix $N_{C_\Theta A_\Lambda}$ is full rank for all matrices $\Theta \in J[0,1]^{p \times n}$, $\Lambda \in J[0,1]^{n \times n}$ and so system (2) is observable for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}, \Omega \in J[0,1]^{p \times m}$; see [4]. Therefore, system (1) is observable. $\square$

4 Criteria-based controllability and observability

Let $S = [s_{ij}]_{m \times n}$ and $Y = [y_{ij}]_{m \times n}$ be real matrices. The absolute of matrix is denoted by $|S| = [|s_{ij}|]_{m \times n}$, and also $S \le Y$ if and only if $s_{ij} \le y_{ij}$ for $i = 1, \ldots, m, \ j = 1, \ldots, n$. The pseudoinverse of S, is a real $n \times m$ matrix

as S^+ such that SS^+ and S^+S are symmetric matrices and $SS^+S = S$ and $S^+SS^+ = S^+$. Suppose that S is full rank and that m is greater than or equal to n. Then $S^+ = (S^TS)^{-1}S^T$ and S^+S is a unit $n \times n$ matrix. If m is smaller than or equal to n, then $S^+ = S^T(SS^T)^{-1}$ and SS^+ is a unit $m \times m$ matrix; see [12]. The singular values of the matrix S are as arithmetic square roots of the joint eigenvalues of the matrices S^TS and SS^T. Also $\sigma_{\min}(S)$ and $\sigma_{\max}(S)$ are respectively the smallest and greatest singular values of the matrix S. Indeed $\rho(\cdot)$ is the spectral radius of a square matrix.

Let $\mathsf{S} = [s^0, s^1]$ be an interval of real numbers. The midpoint of the interval, the radius of the interval, and the absolute of the interval are denoted by $mid\mathsf{S} = \frac{s^0+s^1}{2}$, $rad\mathsf{S} = \frac{s^1-s^0}{2}$ and $|\mathsf{S}| = \max\{|s^0|, |s^1|\}$, respectively. The operations of an interval as, midpoint, radius, and absolute value are element wisely extended for interval matrices. The matrices $\mathcal{S}_{mid}$ and $\mathcal{S}_{rad}$ are called the midpoint matrix and the radius matrix of an interval matrix $\mathcal{S}$, respectively.

Theorem 1. An interval $m \times n$ matrix $\mathcal{S}$, $m \geqslant n$, is full rank if and only if the system of inequalities $|\mathcal{S}_{mid}x| \leqslant \mathcal{S}_{rad}|x|$, $x \in \mathbb{R}^n$, has a unique zero solution.

Proof. See [12]. □

The following definitions propose some criteria that discuss the being the full rank of the interval matrices; see [12].

Definition 9. An interval matrix $\mathcal{S}$ is called **midpoint full rank** if and only if the matrix $\mathcal{S}_{mid}$ is full rank and $\rho(|(\mathcal{S}_{mid})^+| \cdot \mathcal{S}_{rad}) < 1$.

Definition 10. An interval matrix $\mathcal{S}$ is called **singular full rank** if and only if the inequality $\sigma_{\max}(rad\mathcal{S}) < \sigma_{\min}(mid\mathcal{S})$ is satisfied.

Definition 11. An interval matrix $\mathcal{S}$ is called **norm full rank** if and only if the matrix $\mathcal{S}_{mid}$ is full rank and the inequality $\| \mathcal{S}_{rad} \| < \| (\mathcal{S}_{mid})^+ \|^{-1}$ is satisfied. $\| \bullet \|$ is an absolute subordinate matrix norm.

Since, considering to be full rank of the compound matrices $M_{A_\Lambda B_\Gamma}$ and $N_{C_\Theta A_\Lambda}$, for all matrices $\Lambda \in J[0,1]^{n \times n}, \Gamma \in J[0,1]^{n \times m}, \Theta \in J[0,1]^{p \times n}$, due to the presence of many parameters in big problems, looks hard and complicated; therefore, using above criteria, may be useful to overcome the mentioned difficulties. Therefore, using the above definitions, the following criteria for the controllability and observability are presented on the basis of being full rank.

Definition 12. System (1) is controllable or observable with the **midpoint criterion** if and only if the interval matrix $\mathcal{M}_{A\mathcal{B}}$ or $\mathcal{N}_{\mathcal{C}A}$ is the midpoint full rank, respectively.

Definition 13. System (1) is controllable or observable with the **singular criterion** if and only if the interval matrix $\mathcal{M}_{AB}$ or $\mathcal{N}_{CA}$ is the singular full rank, respectively.

Definition 14. System (1) is controllable or observable with the **norm criterion** if and only if the interval matrix $\mathcal{M}_{AB}$ or $\mathcal{N}_{CA}$ is norm full rank, respectively.

Of course, here the relationship between the controllability and observability in general and their relationship with the specified criteria for system (1) is not considered, but, the purpose of the next researches can be to connect these two issues.

5 Numerical examples

Some numerical examples have been presented.

Example 1. Olegovna [?] proposed the following interval optimal control problem:

$$\min \frac{1}{2} \mid x - x_d \mid^2 + \frac{N}{2} \mid u \mid^2,$$

$$s.t. \ aEx(t) = u,$$

$$0 < a_1 \leqslant a \leqslant a_2,$$

where E is a second derivative operator. The state space description of this problem is in the following form:

$\dot{x} = \mathcal{A}x(t) + \mathcal{B}(t)$ such that $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ [\frac{1}{a_2}, \frac{1}{a_1}] \end{bmatrix}$.

By Proposition 1, we have

$A_\Lambda = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_\Gamma = \begin{bmatrix} 0 \\ \frac{1}{a_2} + \gamma_{21}(\frac{1}{a_1} - \frac{1}{a_2}) \end{bmatrix}$, $0 \leqslant \gamma_{21} \leqslant 1$,

$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} 0 & \frac{1}{a_2} + \gamma_{21}(\frac{1}{a_1} - \frac{1}{a_2}) \\ \frac{1}{a_2} + \gamma_{21}(\frac{1}{a_1} - \frac{1}{a_2}) & 0 \end{bmatrix}$,

$det(M_{A_\Lambda B_\Gamma}) = -(\frac{1}{a_2} + \gamma_{21}(\frac{1}{a_1} - \frac{1}{a_2})^2)$, then $det(M_{A_\Lambda B_\Gamma}) \neq 0$.

Therefore, the matrix $M_{A_\Lambda B_\Gamma}$ is full rank for all matrices $\Lambda \in J[0,1]^{2\times2}, \Gamma \in J[0,1]^{2\times1}$ and so the interval compound matrix $\mathcal{M}_{AB}$ is full rank and from Proposition 2 it can be concluded that the system is controllable.

Example 2. Consider the following linear system with interval coefficients only in the state variables that is proposed by Ismail and Bandyopadhyay [8]:

$\mathcal{A} = \begin{bmatrix} [5,6] & [-4,-2] \\ [-4,-2] & [9,10] \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -5 \end{bmatrix}$,

$$A_\Lambda = \begin{bmatrix} 5 + \lambda_{11} & -4 + 2\lambda_{12} \\ -4 + 2\lambda_{21} & 9 + \lambda_{22} \end{bmatrix}, \ B_\Gamma = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ C_\Theta = \begin{bmatrix} 3 & -5 \end{bmatrix},$$

$$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} 1 & -3 + \lambda_{11} + 4\lambda_{12} \\ 2 & 14 + 2\lambda_{21} + 2\lambda_{22} \end{bmatrix},$$

$$N_{C_\Theta A_\Lambda} = \begin{bmatrix} 3 & -5 \\ 35 + 3\lambda_{11} - 10\lambda_{21} & -57 + 6\lambda_{12} - 5\lambda_{22} \end{bmatrix},$$

$det(M_{A_\Lambda B_\Gamma}) = 20 + 2\lambda_{21} - 2\lambda_{11} - 8\lambda_{12}$. The minimum and maximum values of the $det(M_{A_{(\Lambda)}B_{(\Gamma)}})$ are 10 and 24. Therefore, the matrix $M_{A_\Lambda B_\Gamma}$ is full rank for all matrices $\Lambda \in J[0,1]^{2\times 2}$, $\Gamma \in J[0,1]^{2\times 1}$. Therefore the interval compound matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank, and finally the system is controllable.

Now the controllability of system is considered based on the criteria that have been defined in Section 4. For this purpose, after calculating the necessary quantities as

$(\mathcal{M}_{\mathcal{AB}})_{mid} = \begin{bmatrix} 1 & -0.5 \\ 2 & 16 \end{bmatrix}, (\mathcal{M}_{\mathcal{AB}})_{rad} = \begin{bmatrix} 0 & 2.5 \\ 0 & 2 \end{bmatrix}, \rho(|(\mathcal{M}_{\mathcal{AB}})_{mid}|^+ \cdot (\mathcal{M}_{\mathcal{AB}})_{rad}) =$
0.4118, $\sigma_{\min}(\mathcal{M}_{\mathcal{AB}})_{mid} = 1.054, \sigma_{\max}(\mathcal{M}_{\mathcal{AB}})_{rad} = 3.202 \parallel (\mathcal{M}_{\mathcal{AB}})_{mid}^+ \parallel^{-1} =$
1.0303, $\parallel (\mathcal{M}_{\mathcal{AB}})_{rad} \parallel = 2.5$, by Definitions 12– 14, one can consider that the system is controllable with the midpoint criteria. Also Ismail and Bandyopadhyay [8] indicated that this system is controllable.

Moreover $det(N_{C_\Theta A_\Lambda}) = 4 + 18\lambda_{12} + 15\lambda_{11} - 15\lambda_{22} - 50\lambda_{21}$ and the minimum and maximum values of $det(N_{C_\Theta A_\Lambda})$ are -61 and 37. Therefore, there is a matrix $\Lambda \in J[0,1]^{2\times 2}$ such that the matrix $N_{C_\Theta A_\Lambda}$ is not full rank and the interval compound matrix $\mathcal{N}_{\mathcal{CA}}$ is not full rank. Thus the system is not observable. Hence $(\mathcal{N}_{\mathcal{CA}})_{mid} = \begin{bmatrix} 3 & -5 \\ 31.5 & -56.5 \end{bmatrix}$ and $(\mathcal{N}_{\mathcal{CA}})_{rad} = \begin{bmatrix} 0 & 0 \\ 6.5 & 5.5 \end{bmatrix}$.
Therefore, $\rho(|((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \cdot (\mathcal{N}_{\mathcal{CA}})_{rad}) = 4.0833$,
$\sigma_{\min}(\mathcal{N}_{\mathcal{CA}})_{mid} = 0, \ \sigma_{\max}(\mathcal{N}_{\mathcal{CA}})_{rad} = 8.515,$
$\parallel ((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \parallel^{-1} = 0.1951, \ \parallel (\mathcal{N}_{\mathcal{CA}})_{rad} \parallel = 12.$
Then, this system is not observable with all criteria. Also Ismail and Bandyopadhyay [8] indicated that this system is not observable.

Example 3. Consider the following linear system with interval coefficients only in the state variables, which is proposed by Ismail and Bandyopadhyay [8]:

$$\mathcal{A} = \begin{bmatrix} [-8, -4] & [2,3] \\ [2,3] & [5,7] \end{bmatrix}, \ B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

$$A_\Lambda = \begin{bmatrix} -8 + 4\lambda_{11} & 2 + \lambda_{12} \\ 2 + \lambda_{21} & 5 + 2\lambda_{22} \end{bmatrix}, \ B_\Gamma = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ C_\Theta = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

$$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} 4 & -34 + 16\lambda_{11} - \lambda_{12} \\ -1 & 3 + 4\lambda_{21} - 2\lambda_{22} \end{bmatrix},$$

$$N_{C_\Theta A_\Lambda} = \begin{bmatrix} 2 & 1 \\ -14 + 8\lambda_{11} + \lambda_{21} & 9 + 2\lambda_{12} + 2\lambda_{22} \end{bmatrix},$$

$det(M_{A_\Lambda b_\Gamma}) = -22 + 16\lambda_{21} - 8\lambda_{22} + 16\lambda_{11} - \lambda_{12}$. The minimum and maximum values of $det(M_{A_\Lambda B_\Gamma})$ are -31 and 10. Hence there is a matrix $\Lambda \in J[0,1]^{2\times 2}$ such that the matrix $M_{A_\Lambda b_\Gamma}$ is not full rank and the interval compound ma-

trix $\mathcal{M}_{\mathcal{AB}}$ is not full rank. Therefore, the system is not controllable.

$(\mathcal{M}_{\mathcal{AB}})_{mid} = \begin{bmatrix} 4 & -26.5 \\ -1 & 4 \end{bmatrix}$, $(\mathcal{M}_{\mathcal{AB}})_{rad} = \begin{bmatrix} 0 & 8.5 \\ 0 & 3 \end{bmatrix}$. Therefore

$\rho(|((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \cdot (\mathcal{M}_{\mathcal{AB}})_{rad}) = 1.9524$,

$\sigma_{\min}((\mathcal{M}_{\mathcal{AB}})_{mid}) = 0.387$, $\sigma_{\max}((\mathcal{M}_{\mathcal{AB}})_{rad}) = 9.014$,

$\| (\mathcal{M}_{\mathcal{AB}})_{mid})^+ \|^{-1} = 0.3443$, $\| (\mathcal{M}_{\mathcal{AB}})_{rad} \| = 8.5$.

Then this system is not controllable with all criteria. Ismail and Bandyopadhyay [8] indicated this system is not controllable.

Also $det(N_{C_\Theta A_\Lambda}) = 10 + 4\lambda_{12} + 4\lambda_{22} - 8\lambda_{11} - \lambda_{21}$ and so The minimum and maximum values of $det(N_{C_\Theta A_\Lambda})$ are 1 and 18. Then, the matrix $N_{C_\Theta A_\Lambda}$ is full rank for all matrices $\Theta \in J[0,1]^{1\times2}, \Lambda \in J[0,1]^{2\times2}$. Then, the interval compound matrix $\mathcal{N}_{\mathcal{CA}}$ is full rank. Therefore, This system is observable.

Moreover, $(\mathcal{N}_{\mathcal{CA}})_{mid} = \begin{bmatrix} 2 & 1 \\ -9.5 & 11 \end{bmatrix}$, $(\mathcal{N}_{\mathcal{CA}})_{rad} = \begin{bmatrix} 0 & 0 \\ 4.5 & 2 \end{bmatrix}$. Therefore

$\rho(|((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \cdot (\mathcal{N}_{\mathcal{CA}})_{rad}) = 0.26998$,

$\sigma_{\min}((\mathcal{N}_{\mathcal{CA}})_{mid}) = 2.166$, $\sigma_{\max}((\mathcal{N}_{\mathcal{CA}})_{rad}) = 4.924$,

$\| ((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \|^{-1} = 2.625$, $\| (\mathcal{N}_{\mathcal{CA}})_{rad} \| = 6.5$.

Then, this system is observable with midpoint criterion, and Ismail and Bandyopadhyay [8] also indicated that this system is observable.

Example 4. Yang [15] proposed a linear system with interval coefficients as follows:

$A \in \begin{bmatrix} 1 \pm 0.01 & -1 \pm 0.01 \\ 0 \pm 0.01 & 4 \pm 0.01 \end{bmatrix}$, $B \in \begin{bmatrix} 1 \pm 0.01 \\ 1 \pm 0.01 \end{bmatrix}$. Therefore

$\mathcal{A} = \begin{bmatrix} [0.99, 1.01] & [-1.01, -0.99] \\ [-0.01, 0.01] & [3.99, 4.01] \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} [0.99, 1.01] \\ [-1.01, -0.99] \end{bmatrix}$,

$A_\Lambda = \begin{bmatrix} 0.99 + 0.02\lambda_{11} & -1.01 + 0.02\lambda_{12} \\ -0.01 + 0.02\lambda_{21} & 3.99 + 0.02\lambda_{22} \end{bmatrix}$, $B_\Gamma = \begin{bmatrix} 0.99 + 0.02\gamma_{11} \\ -1.01 + 0.02\gamma_{21} \end{bmatrix}$,

$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ such that

$$m_{11} = 0.99 + 0.02\gamma_{11}, \quad m_{21} = -1.01 + 0.02\gamma_{21},$$
$$m_{12} = 2.0002 + 0.0198\gamma_{11} + 0.0198\lambda_{11} + 0.0004\lambda_{11}\gamma_{11}$$
$$- 0.0202\gamma_{21} - 0.0202\lambda_{12} + 0.0004\lambda_{12}\gamma_{21},$$
$$m_{22} = -4.0398 - 0.0002\gamma_{11} + 0.0198\lambda_{12} + 0.0004\lambda_{21}\gamma_{11}$$
$$+ 0.0798\gamma_{21} - 0.0202\lambda_{22} + 0.0004\lambda_{22}\gamma_{21},$$

Using computational algebra methods, it can be shown that the determinant of the matrix $M_{A_\Lambda B_\Gamma}$ will not be zero, and thus the system is controllable. Here we review the controllability based on the defined criteria. Thus

$(\mathcal{M}_{\mathcal{AB}})_{mid} = \begin{bmatrix} 1 & 2.0002 \\ 1 & 4.0001 \end{bmatrix}$, $(\mathcal{M}_{\mathcal{AB}})_{rad} = \begin{bmatrix} 0.01 & 0.04 \\ 0.01 & 0.0601 \end{bmatrix}$.

Midpoint criterion: The matrix $(\mathcal{M}_{\mathcal{AB}})_{mid}$ is full rank and

$\rho(|((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \cdot (\mathcal{M}_{\mathcal{AB}})_{rad}) = 0.0788 < 1$. The system is controllable in

accordance with the midpoint criterion.

Singular criterion: $\sigma_{\min}((\mathcal{M}_{\mathcal{AB}})_{mid}) = 0.1833$, $\sigma_{\max}((\mathcal{M}_{\mathcal{AB}})_{rad}) = 0.0054$, $\sigma((\mathcal{M}_{\mathcal{AB}})_{rad}) < \sigma_{\min}((\mathcal{M}_{\mathcal{AB}})_{mid})$. Therefore, the system is controllable in accordance with the singular criterion.

Norm criterion: The matrix $(\mathcal{M}_{\mathcal{AB}})_{mid}$ is full rank and $\| ((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \|^{-1} = 0.333$, $\| (\mathcal{M}_{\mathcal{AB}})_{rad} \| = 0.0701$, and $\| (\mathcal{M}_{\mathcal{AB}})_{rad} \| < \| ((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \|^{-1}$. Then the system is controllable in accordance with the norm criterion.

Therefore, this system is controllable in accordance with all criteria. Yang [15] also indicated that this system is controllable.

Example 5. Consider a linear system with following interval coefficients:

$\mathcal{A} = \begin{bmatrix} [5,7] & [2,6] \\ [-9,-7] & [5,10] \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} [6,10] \\ [3,7] \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} [-8,-5] & [1,6] \end{bmatrix}$,

$A_\Lambda = \begin{bmatrix} 5+2\lambda_{11} & 2+4\lambda_{12} \\ -9+2\lambda_{21} & 5+5\lambda_{22} \end{bmatrix}$, $B_\Gamma = \begin{bmatrix} 6+4\gamma_{11} \\ 3+4\gamma_{21} \end{bmatrix}$,

$C_\Theta = \begin{bmatrix} -8+3\theta_{11} & 1+5\theta_{12} \end{bmatrix}$,

$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ such that

$m_{11} = 6+4\gamma_{11}$, $m_{21} = 3+4\gamma_{21}$,

$m_{12} = 36+20\gamma_{11}+12\lambda_{11}+8\lambda_{11}\gamma_{11}+8\gamma_{21}+12\lambda_{12}+16\gamma_{21}\lambda_{12}$,

$m_{22} = -39-36\gamma_{11}+12\lambda_{21}+8\lambda_{21}\gamma_{11}+20\gamma_{21}+15\lambda_{22}+20\lambda_{22}\gamma_{21}$.

As the previous example, using computational algebra methods, it can be shown that the determinant of the matrix $M_{A(\Lambda)B(\Gamma)}$ may be zero and thus the system is not controllable. To check the controllability based on the given criteria, we have the following:

Midpoint criterion: The matrix $(\mathcal{M}_{\mathcal{AB}})_{mid}$ is full rank and $\rho(|((\mathcal{M}_{\mathcal{AB}})_{mid})^+| \cdot (\mathcal{M}_{\mathcal{AB}})_{rad}) = 1.4393 > 1$.

Singular criterion: $\sigma_{\min}((\mathcal{M}_{\mathcal{AB}})_{mid}) = 7.162$, $\sigma_{max}((\mathcal{M}_{\mathcal{AB}})_{rad}) = 64.063$, $\sigma_{max}((\mathcal{M}_{\mathcal{AB}})_{rad}) > \sigma_{\min}((\mathcal{M}_{\mathcal{AB}})_{mid})$.

Norm criterion: The matrix $(\mathcal{M}_{\mathcal{AB}})_{mid}$ is full rank and $\| ((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \|^{-1} = 5.7231$, $\| (\mathcal{M}_{\mathcal{AB}})_{rad} \| = 53.5$, and $\| (\mathcal{M}_{\mathcal{AB}})_{rad} \| > \| ((\mathcal{M}_{\mathcal{AB}})_{mid})^+ \|^{-1}$. The interval compound matrix $\mathcal{M}_{\mathcal{AB}}$, in accordance with any of the criteria, is not full rank, so, this system is not controllable based on the given criteria.

Also $N_{C_\Theta A_\Lambda} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$ such that

$n_{11} = -8+3\theta_{11}$, $n_{12} = 1+5\theta_{12}$,

$n_{21} = -49-16\lambda_{11}+15\theta_{11}+6\theta_{11}\lambda_{11}+2\lambda_{21}-45\theta_{12}+10\theta_{12}\lambda_{21}$,

$n_{22} = -11-32\lambda_{12}+6\theta_{11}+12\theta_{11}\lambda_{12}+5\lambda_{22}+25\theta_{12}+25\theta_{12}\lambda_{22}$,

The matrix $N_{C(\Theta)A(\Lambda)}$ may be zero and thus the system may not be observable. To check the observability by the given criteria, we have

$(\mathcal{N}_{\mathcal{CA}})_{mid} = \begin{bmatrix} -6.5 & 3.5 \\ 71 & 3.5 \end{bmatrix}$, $(\mathcal{N}_{\mathcal{CA}})_{rad} = \begin{bmatrix} 1.5 & 2.5 \\ 39 & 46.5 \end{bmatrix}$.

Midpoint criterion: The matrix $(\mathcal{N}_{\mathcal{CA}})_{mid}$ is full rank and $\rho(|((\mathcal{N}_{\mathcal{CA}})_{mid})^+| \cdot (\mathcal{N}_{\mathcal{CA}})_{rad}) = 2.2535 > 1$.

Singular criterion: $\sigma_{\min}((\mathcal{N}_{\mathcal{CA}})_{mid}) = 0.118$, $\sigma_{max}((\mathcal{N}_{\mathcal{CA}})_{rad}) = 60.7577$.

$\sigma_{max}((\mathcal{N}_{\mathcal{CA}})_{rad}) > \sigma_{\min}((\mathcal{N}_{\mathcal{CA}})_{mid})$.

Norm criterion: The matrix $(\mathcal{N}_{\mathcal{CA}})_{mid}$ is full rank and $\| ((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \|^{-1}=$ 5.7231, $\| (\mathcal{N}_{\mathcal{CA}})_{mid} \|= 53.5$ and $\| (\mathcal{N}_{\mathcal{CA}})_{rad} \|>\| ((\mathcal{N}_{\mathcal{CA}})_{mid})^+ \|^{-1}$.

The interval compound matrix $\mathcal{N}_{\mathcal{CA}}$, in accordance with all criteria, is not full rank, so, this system is not observable based on criteria.

Example 6. Cheng and Zhang in [2] presented a system with the following coefficient matrices, and they indicated that this system is controllable using the feedback:

$$A \in \begin{bmatrix} 0 \pm 0.05 & 0 & 0 \\ 0 & 1 \pm 0.04 & 1 \pm 0.03 \\ 0 & 0 \pm 0.08 & 0 \pm 0.4 \end{bmatrix} , B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} . \text{ Therefore,}$$

$$\mathcal{A} = \begin{bmatrix} [-0.05, 0.05] & 0 & 0 \\ 0 & [0.96, 1.04] & [0.97, 1.03] \\ 0 & [-0.08, 0.08] & [-0.4, 0.4] \end{bmatrix} , \mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$A_\Lambda = \begin{bmatrix} -0.05 + 0.1\lambda_{11} & 0 & 0 \\ 0 & 0.96 + 0.08\lambda_{22} & 0.97 + 0.06\lambda_{23} \\ 0 & -0.08 + 0.16\lambda_{32} & -0.4 + 0.8\lambda_{33} \end{bmatrix} , B_\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$A_\Lambda^2 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ such that}$$

$a_{11} = (-0.05 + 0.1\lambda_{11})^2$,

$a_{22} = 0.844 + 0.1536\lambda_{22} + 0.0064\lambda_{22}^2 - 0.0048\lambda_{23} + 0.1552\lambda_{32} + 0.0096\lambda_{32}\lambda_{23}$,

$a_{23} = 0.8825 + 0.0336\lambda_{23} + 0.0776\lambda_{22} + 0.0048\lambda_{22}\lambda_{23} + 0.776\lambda_{33} + 0.048\lambda_{23}\lambda_{33}$,

$a_{32} = -0.0448 + 0.512\lambda_{32} - 0.0064\lambda_{22} - 0.064\lambda_{33} + 0.048\lambda_{22}\lambda_{32} + 0.128\lambda_{33}\lambda_{32}$,

$a_{33} = 0.0824 - 0.0048\lambda_{23} + 0.1552\lambda_{32} + 0.0096\lambda_{32}\lambda_{23} + 0.64\lambda_{33}^2 - 0.64\lambda_{33}$.

Therefore,

$$M_{A_\Lambda B_\Gamma} = \begin{bmatrix} 1 & 0 & -0.05 + 0.1\lambda_{11} & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0.97 + 0.06\lambda_{23} & 0 & a_{23} \\ 0 & 1 & 0 & -0.4 + 0.8\lambda_{33} & 0 & a_{33} \end{bmatrix} \text{ and}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{bmatrix}$ is a square submatrix of the matrix $M_{A_\Lambda B_\Gamma}$, so that its determinant is $-a_{23}$. The minimum and maximum of $-a_{23}$ are 0.8825 and 1.8225, respectively. As a result, the matrix $M_{A(\Lambda)B(\Gamma)}$ is full rank for all matrices $\Lambda \in J[0,1]^{3\times3}$, $\Gamma \in J[0,1]^{3\times2}$, and so the system is controllable. The controllability of this system is determined without using the feedback and this is the advantage of the proposed method.

With respect to these examples, it can be concluded that the controllability and observability of an interval control system do not guarantee the controllability and observability based on the defined criteria. If a system is controllable or observable with all criteria, then this system may be controllable or observable. Whenever, a system is not controllable or observable with all criteria, then that system may not be controllable or observable.

6 Conclusion

In this paper, using the parametric representation of interval matrices in the time-invariant (continuous time) linear system with interval coefficients, the classical approach of determining controllability and observability of these systems by reproducing compound matrices was discussed. The approach helps to investigate the controllability and observability for general MIMO systems without considering additional restrictions in coefficients matrices. Undoubtedly, the computations of checking to be full rank of compound matrices may be complicated and so by different concepts in to be full rank of matrices, some criteria for evaluating these concepts were given. The relationship between the actual controllability and observability of interval linear systems and what is presented based on the given criteria may be the basis of interesting researches in this area.

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